Additive Combinatorics Notes

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Notes from my Additive Combinatorics class at Cambridge with Julia Wolf. Any mistake is with very high certainty mine.

Chapter 1: Fourier-Analytic Techniques

Lecture 1

Let $G = \mathbb{F}_p^n$ where p is a small fixed prime ($p = 2, 3, 5$) and n is large ($n \to \infty$). Notation: Given a finite set B and any function $f : B \to \mathbb{C}$, write $\mathbb{E}_{x\in B} f(x) := \dfrac{1}{|B|} \sum_{x\in B} f(x)$. Write $\omega = e^{2\pi i/p}$ for a p-th root of unity. Note $\sum_{a\in\mathbb{F}_p} \omega^a = 0$.

Definition 1.1 Given $f : \mathbb{F}_p^n \to \mathbb{C}$, define its **Fourier transform** $\widehat{f} : \mathbb{F}_p^n \to \mathbb{C}$ by $\widehat{f}(t) = \mathbb{E}_{x\in\mathbb{F}_p^n} f(x)\omega^{x\cdot t}$ for all $t \in \mathbb{F}_p^n$.

It's easy to verify the **inversion formula**: $f(x) = \sum_{t\in\mathbb{F}_p^n} \widehat{f}(t)\omega^{-x\cdot t}$. Indeed,

$$\sum_{t\in\mathbb{F}_p^n} \widehat{f}(t)\omega^{-x\cdot t} = \sum_{t\in\mathbb{F}_p^n} (\mathbb{E}_y f(y)\omega^{-y\cdot t})\omega^{-x\cdot t} = \mathbb{E}_y f(y) \underbrace{\sum_{t\in\mathbb{F}_p^n} \omega^{(y-x)\cdot t}}_{p^n 1_{\{y=x\}}} = f(x).$$

Notation: Given a subset A of a finite group G, write:

- 1_A for the **characteristic function** of A (or indicator function)
- f_A for the **balanced function** of A. i.e. $f_A(x) = 1_A(x) - \alpha$ where $\alpha = \frac{|A|}{|G|}$.
- μ_A for the **characteristic measure** of A. i.e. $\mu_A(x) = \alpha^{-1} 1_A(x)$.

Note $\mathbb{E}_{x\in G} f_A(x) = 0$ and $\mathbb{E}_{x\in G}\mu_A(x) = 1$. Note that given $A \subset \mathbb{F}_p^n$, we have $\widehat{1_A}(f) = \mathbb{E}_{x\in\mathbb{F}_p^n} 1_A(x)\omega^{x\cdot t}$. So $\widehat{1_A}(0) = \mathbb{E}_{x\in\mathbb{F}_p^n} 1_A(x) = \alpha$. Writing $-A = \{-a : a \in A\}$, we have

$$\widehat{1_{-A}}(t) = \mathbb{E}_{x\in\mathbb{F}_p^n} 1_{-A}(x)\omega^{x\cdot t} = \mathbb{E}_{x\in\mathbb{F}_p^n} 1_A(-x)\omega^{x\cdot t} = \mathbb{E}_{y\in\mathbb{F}_p^n} 1_A(y)\omega^{-y\cdot t} = \overline{\mathbb{E}_{y\in\mathbb{F}_p^n} 1_A(y)\omega^{y\cdot t}} = \overline{\widehat{1_A}(t)}.$$

Example 1.2 Let $V \leq \mathbb{F}_p^n$. Then $\widehat{1_V}(t) = \mathbb{E}_{x\in\mathbb{F}_p^n} 1_V(x)\omega^{x\cdot t} = \dfrac{|V|}{p^n} 1_{\{x\cdot t = 0\ \forall x\in V\}}(t) = \dfrac{|V|}{p^n} 1_{V^\perp}(t)$. So $\widehat{\mu_V}(t) = 1_{V^\perp}(t)$.

Let's look at the opposite. Instead of having a lot of structure in the subvectorspace, we'll go to the other extreme with R a random set.

Example 1.3: Let $R \subset \mathbb{F}_p^n$ be such that each $x \in \mathbb{F}_p^n$ lies in R independently with probability $1/2$. Then with high probability $\sup_{t\neq 0} |\widehat{1_R}(t)| = O(\sqrt{\frac{\log(p^n)}{p^n}})$.

We'll show this in the first example sheet using a **Chernoff-type bound**: Given $\mathbb{C}$-valued independent random variables $X_1, \ldots, X_n$ with mean 0, for all $\theta \geq 0$ we have $\mathbb{P}[|\sum x_i| \geq \theta\sqrt{\sum \|x_i\|^2_{L^\infty(\mathbb{P})}}] \leq 4\exp(-\theta^2/4)$.

Example 1.4. Let $Q = \{x \in \mathbb{F}_p^n : x \cdot x = 0\}$. Then $|Q| = (\frac{1}{p} + O(p^{-n}))p^n$ and $\sup_{t\neq 0} |\widehat{1_Q}(t)| = O(p^{-n/2}) \longrightarrow$ Example Sheet 1.

Notation: Given $f, g : \mathbb{F}_p^n \to \mathbb{C}$, write $\langle f, g\rangle := \mathbb{E}_{x\in\mathbb{F}_p^n} f(x)\overline{g(x)}$ and $\langle \widehat{f}, \widehat{g}\rangle := \sum_{t\in\mathbb{F}_p^n} \widehat{f}(t)\overline{\widehat{g}(t)}$.

Consequently, $\|f\|_2^2 = \mathbb{E}_x |f(x)|^2$ while $\|\widehat{f}\|_2^2 = \sum_{t\in\mathbb{F}_p^n} |\widehat{f}(t)|^2$.

Lemma : 1.5

The following hold for all $f, g : \mathbb{F}_p^n \to \mathbb{C}$:

(i) $\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle$ (**Plancharel's identity**)

(ii) $\|f\|_2 = \|\widehat{f}\|_2$ (**Parseval's identity** or energy conservation)

Proof. Exercise. □

Definition 1.6. Let $\rho > 0$ and $f : \mathbb{F}_p^n \to \mathbb{C}$. Define the **$\rho$-large spectrum** of the f to be

$$\mathrm{Spec}_\rho(f) = \{t \in \mathbb{F}_p^n : |\widehat{f}(t)| \geq \rho\|f\|_1\}.$$

Lemma : 1.8

For all $\rho > 0$, $|\mathrm{Spec}_\rho(f)| \leq \rho^{-2}\frac{\|f\|_2^2}{\|f\|_1^2}$.

Proof. $\|f\|_2^2 = \|\widehat{f}\|_2^2 \geq \sum_{t\in\mathrm{Spec}_\rho(f)} |\widehat{f}(t)|^2 \geq |\mathrm{Spec}_\rho(f)|(\rho\|f\|_1)^2$. □

Lecture 2

Definition 1.9. Given $f, g : \mathbb{F}_p^n \to \mathbb{C}$, define their **convolution** $f * g : \mathbb{F}_p^n \to \mathbb{C}$ by $f * g(x) := \mathbb{E}_{y\in\mathbb{F}_p^n} f(y)g(x-y)$.

Example 1.10. Given $A, B \subset \mathbb{F}_p^n$.

$$1_A * 1_B(x) = \mathbb{E}_{y\in\mathbb{F}_p^n} 1_A(y)1_B(x-y) = \frac{1}{p^n}|A \cap (x - B)|$$

So
$$1_A * 1_B(x) = \frac{1}{p^n}\#\text{ways } x \text{ can be written as } x = a + b \text{ with } a \in A, b \in B.$$
In particular, the support of $1_A * 1_B$ is the **sum set** $A + B = \{a + b : a \in A, b \in B\}$.

Lemma : 1.11

Given $f, g : \mathbb{F}_p^n \to \mathbb{C}$: $\widehat{f * g}(t) = \widehat{f}(t)\widehat{g}(t)$ for $t \in \mathbb{F}_p^n$.

Proof. □

Example 1.12. $\|\widehat{f}\|_4^4 = \underset{x+y=w+z}{\mathbb{E}} f(x)f(y)\overline{f(w)f(z)}$.
We'll prove this in the first example sheet.

Lemma : 1.13 (Bogolyubov)

Given $A \subset \mathbb{F}_p^n$ of density $\alpha > 0$, there exists a subspace $V \leq \mathbb{F}_p^n$ of codimension at most $2\alpha^{-2}$ such that $A + A - A - A \supset V$.

Proof. Observe that $A + A - A - A = \mathrm{supp}(\underbrace{1_A * 1_A * 1_{-A} * 1_{-A}}_{g(x)})$. We wish to find $V \leq \mathbb{F}_p^n$ such that $g(x) > 0$ for all $x \in V$. Let $K = \mathrm{Spec}_\rho(1_A)$ with ρ to be determined. Let $V = \langle K \rangle^\perp$. By lemma 1.8 we have $|K| \leq \rho^{-2}\alpha^{-1}$ and therefore $\mathrm{codim}(V) \leq |K| \leq \rho^{-2}\alpha^{-1}$.

$$\begin{aligned} g(x) &= \sum_{t \in \mathbb{F}_p^n} \widehat{g}(t)\omega^{-x\cdot t} = \sum_{t \in \mathbb{F}_p^n} (\widehat{1_A}(t))^2(\widehat{1_{-A}}(t))^2\omega^{-x\cdot t} \\ &= \sum_{t \in \mathbb{F}_p^n} |\widehat{1_A}(t)|^4\omega^{-x\cdot t} = \alpha^4 + \sum_{t \neq 0} |\widehat{1_A}(t)|^4\omega^{-x\cdot t} \\ &= \alpha^4 + \underbrace{\sum_{t \in K\setminus\{0\}} |\widehat{1_A}(t)|^4\omega^{-x\cdot t}}_{(1)} + \underbrace{\sum_{t \notin K} |\widehat{1_A}(t)|^4\omega^{-x\cdot t}}_{(2)} \end{aligned}$$

Clearly $(1) \geq 0$ since $x \cdot t = 0$ for all $t \in K$ and $x \in V$.
On the other hand,

$$|(2)| \leq \sum_{t \notin K} |\widehat{1_A}(t)|^4 \leq \sup_{k \notin K} |\widehat{1_A}(t)|^2 \sum_t |\widehat{1_A}(t)|^2.$$

By Parseval's identity we get

$$|(2)| \leq (\rho\alpha)^2\|1_A\|_2^2 = \rho^2\alpha^3.$$

So pick ρ such that $\rho^2\alpha^3 \leq \alpha^4/2$ (for example $\rho = \sqrt{\alpha/2}$) This gives $\mathrm{codim}(V) \leq 2\alpha^{-2}$. □

Example 1.14. The set $A = \{x \in \mathbb{F}_2^n : |x| \geq \frac{n}{2} + \frac{\sqrt{n}}{2}\}$ has density at least $1/4$ and there is no coset C of any subspace of codimension $\sqrt{n}$ such that $C \subset A + A$. We'll see this on example sheet 1.

Lemma : 1.15

Let $A \subset \mathbb{F}_p^n$ of density α be such that $\exists t \neq 0$ in $\mathrm{Spec}_\rho(1_A)$. Then there exists $V \leq \mathbb{F}_p^n$ of codimension 1 and exists $x \in \mathbb{F}_p^n$ such that $|A \cap (x + V)| \geq \alpha(1 + \rho/2)|V|$.

Proof. Let $t \neq 0$ be such that $|\widehat{1_A}(t)| \geq \rho\alpha$, and let $V = \langle t \rangle^\perp$. Write $v_j + V$ for $j \in [p] = \{1, 2, \dots, p\}$ for the cosets of V such that $v_j + V = \{x \in \mathbb{F}_p^n : x \cdot t = j\}$. Then $\widehat{1_A}(t) = \widehat{f_A}(t) = \mathop{\mathbb{E}}_{x \in \mathbb{F}_p^n}(1_A(x) - \alpha)\omega^{x \cdot t} = \mathop{\mathbb{E}}_{j \in [p]} \underbrace{\mathop{\mathbb{E}}_{x \in v_j + V}(1_A(x) - \alpha)}_{=:a_j}\omega^j$ where $a_j = \dfrac{|A \cap (v_j + V)|}{|v_j + V|} - \alpha$. By the triangle inequality $\mathop{\mathbb{E}}_{j \in [p]} |a_j| \geq \rho\alpha$. Since $\mathop{\mathbb{E}}_{j \in [p]} a_j = 0$, $\mathop{\mathbb{E}}_{j \in [p]}(a_j + |a_j|) \geq \rho\alpha$ implies that there exists $j \in [p]$ such that $a_j + |a_j| \geq \rho\alpha$ therefore $a_j \geq \rho\dfrac{\alpha}{2}$. $\square$

Lecture 3

Lemma : 1.16

Let $p \geq 3$ and $A \subset \mathbb{F}_p^n$ has density $\alpha > 0$. Let A be such that $\sup_{k \neq 0} |\widehat{1_A}(t)| = o(1)$. Then A contains $(\alpha^3 + o(1))(p^n)^2$ 3-term arithmetic progressions.

Notation:

- 3-AP = 3-term arithmetic progression.
- Write $2 \cdot A = \{2a : a \in A\}$. It's important to distinguish this from $2A = A + A = \{a + a' : a, a' \in A\}$.

Proof. The number of 3-APs in A is $(p^n)^2$ times

$$
\begin{aligned}
T_3(1_A, 1_A, 1_A) &= \mathop{\mathbb{E}}_{x,d}[1_A(x)1_A(\overbrace{x + d}^{y})1_A(x + 2d)] \\
&= \mathop{\mathbb{E}}_{x,y}[1_A(x)1_A(y)1_A(2y - x)] = \mathop{\mathbb{E}}_{y}[1_A(y)1_A * 1_A(2y)] \\
&= \langle 1_{2 \cdot A}, 1_A * 1_A \rangle.
\end{aligned}
$$

By Plancharel's identity we get taht this is equal to

$$
\langle \widehat{1_{2 \cdot A}}, \widehat{1_A * 1_A} \rangle = \langle \widehat{1_{2 \cdot A}}, \widehat{1_A} \cdot \widehat{1_A} \rangle = \alpha^3 + \underbrace{\sum_{t \neq 0} \widehat{1_A}(t)^2 \overline{\widehat{1_{2 \cdot A}}(t)}}_{(1)}
$$

. In absolute value, the sum above is

$$|(1)| \leq \sup_{t\neq 0} |\widehat{1_A}(t)| \sum_{t\neq 0} |\widehat{1_A}(t) \cdot \overline{\widehat{1_{2\cdot A}}(t)}|$$

$$|(1)| \leq \sup_{t\neq 0} |\widehat{1_A}(t)| \cdot (\sum_t |\widehat{1_A}(t)|^2)^{1/2} \cdot (\sum_t |\widehat{1_{2\cdot A}}(t)|^2)^{1/2}$$

By Parseval this becomes

$$|(1)| \leq \sup_{t\neq 0} |\widehat{1_A}(t)| \cdot \alpha^{1/2} \cdot \alpha^{1/2}.$$

□

We shall combine these observations to prove the following:

Theorem : 1.17 - Meshulam

Let $A \subset \mathbb{F}_p^n$, $p \geq 3$, be a set containing no nontrivial 3-AP. Then $|A| = O\left(\frac{p^n}{n \log p}\right)$.

Proof. By assumption, $T_3(1_A, 1_A, 1_A) = \frac{\alpha}{p^n}$ but as in lemma 1.16 $T_3(1_A, 1_A, 1_A) = \alpha^3 + \sum_{t\neq 0} \widehat{1_A}(t)^2 \widehat{1_{2\cdot A}}(t)$.

Observation:

Provided that $p^n \geq 2\alpha^{-2}$, we have

$$\left| \frac{\alpha}{p^n} - \alpha^3 \right| \leq \sup_{t\neq 0} |\widehat{1_A}(t)| \alpha$$

That is $\sup_{t\neq 0} |\widehat{1_A}(t)| \geq \frac{\alpha^2}{2}$.
By lemma 1.15 with $\rho = \frac{\alpha}{2}$ there exists $V \leq \mathbb{F}_p^n$ of codimension 1 and $x \in \mathbb{F}_p^n$ such that

$$|A \cap (x + V)| \geq (\alpha + \frac{\alpha^2}{4})|V|.$$

We iterate this observation. Let $A_0 = A$, $V_0 = \mathbb{F}_p^n$, $\alpha_0 = \frac{|A_0|}{|V_0|} = \alpha$.
At step i we are given a set $A_{i-1} \subset V_{i-1}$ of density α_{i-1} with no nontrivial 3-APs.
Provided that $p^{\dim(V_{i-1})} \geq 2\alpha_{i-1}^{-2}$ there exists $V_i \leq V_{i-1}$ of codimension 1 and $x_i \in V_{i-1}$ such that

$$|A_{i-1} \cap (x_i + V_i)| \geq (\alpha_{i-1} + \frac{\alpha_{i-1}^2}{4})|V_{i-1}|.$$

Set $A_i = A_{i-1} - x_i$. This set will be 3-AP-free because we're shifting a 3-AP-free set. Note that $\alpha_i \geq \alpha_{i-1} + \dfrac{\alpha_{i-1}^2}{4}$. Through this iteration, the density of A increases from α to 2α in at most $4\alpha^{-1}$ steps. From 2α to 4α in at most $2\alpha^{-1}$ steps,..., and reaches 1 in at most $4\alpha^{-1}(1 + 1/2 + 1/4 + \dots) = 8\alpha^{-1}$.
The argument must therefore end with $\dim(V_i) \geq n - 8\alpha^{-1}$ at which point we must've had $p^{\dim(V_i)} < 2\alpha_i^{-2} \leq 2\alpha^{-2}$. But we may assume that $\alpha \geq \sqrt{2}p^{-n/4}$ whence $p^{n-8\alpha^{-1}} \leq p^{n/2}$ or $n/2 \leq 8\alpha^{-1}$. $\square$

Lecture 4

Last time we proved that if $A \subset \mathbb{F}_3^n$ contains no non-trivial 3-APs, then $|A| = O(\frac{3^n}{n})$.
The largest known subset of $\mathbb{F}_3^n$ containing no non-trivial 3-APs has size $\geq (2.218)^n$ due to Tyrrell (2022) - we'll return to this later.
From now on, let G be a finite abelian group. G comes equipped with a set of **characters**, i.e. group homomorphisms $\gamma : G \to \mathbb{C}^\times$, which themselves form a group, denoted by $\widehat{G}$, and is referred to as the **dual** of G.
It turns out that if G is finite abelian, then $\widehat{G} = G$.
For instance, if $G \simeq \mathbb{F}_p^n$, then $\widehat{G} = \{\gamma_t : x \mapsto \omega^{x \cdot t}, t \in G\}$. If $G = \mathbb{Z}_p := \mathbb{Z}/p\mathbb{Z}$ (not p-adics!), then $\widehat{G} = \{\gamma_t : x \mapsto \omega^{xt}, t \in G\}$.
Definition 1.18. Given $f : G \to \mathbb{C}$ define its **Fourier transform** $\widehat{f} : \widehat{G} \to \mathbb{C}$ by

$$\widehat{f}(\gamma) = \mathop{\mathbb{E}}_{x \in G} f(x)\gamma(x) \ \ \forall \gamma \in \widehat{G}.$$

It is easy to verify that

$$f(x) = \sum_{\gamma \in \widehat{G}} \widehat{f}(\gamma)\overline{\gamma(x)}.$$

You may also check that definitions 1.6, 1.9; examples 1.3 and 1.10; and lemmas 1.5, 1.8, 1.11 go through in this more general context.
Example 1.19. Let p be a prime, let $L \leq p-1$ be even and consider $J = \left[-\frac{L}{2}, \frac{L}{2}\right] \subset \mathbb{Z}_p$. Then $\forall t \neq 0$ we have $|\widehat{1_J}(t)| \leq \min\{\frac{L+1}{p}, \frac{1}{2|t|}\}$. We'll see this in Example Sheet 1.

Theorem : 1.20. (Roth)

Let $A \subset [N]$ be a set containing no nontrivial 3-APs. Then $|A| = O\left(\dfrac{N}{\log\log N}\right)$.

Lemma : 1.21

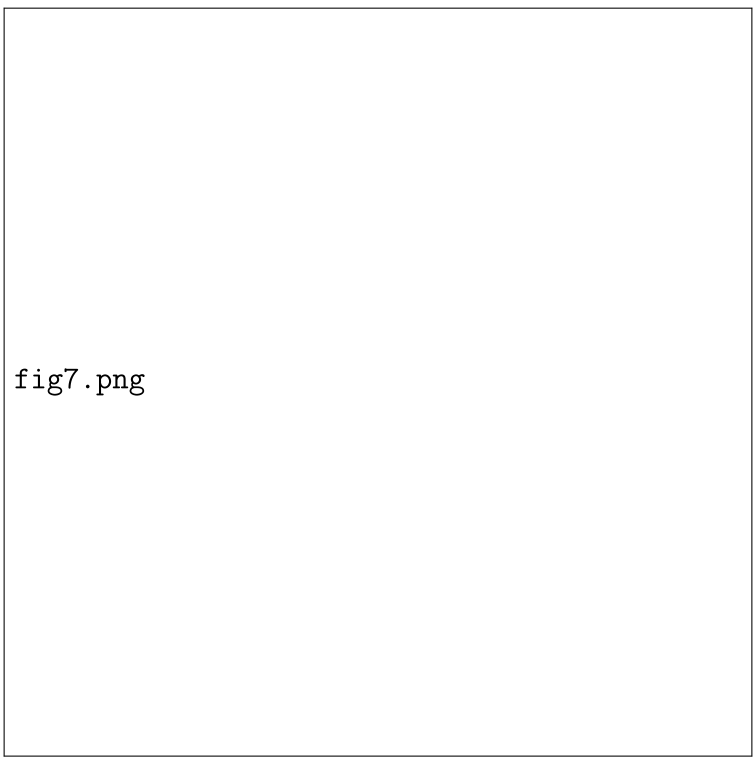
Let $A \subset [N]$ be of density $\alpha > 0$ satisfying $N > 50\alpha^{-2}$, containing no non-trivial 3-APs. Let p be a prime in $\left[\frac{N}{3}, \frac{2N}{3}\right]$ and write $A' = A \cap [p] \subset \mathbb{Z}_p$. Then either:

(i) $\sup_{t \neq 0} |\widehat{1_{A'}}(t)| \geq \dfrac{\alpha^2}{10}$ (where the Fourier coefficient is computed in $\mathbb{Z}_p$), or

(ii) There exists an interval $J \subset [N]$ of length $\geq \dfrac{N}{3}$ such that

$$|A \cap J| \geq \alpha \left(1 + \frac{\alpha}{400}\right) |J|.$$

Proof. We may assume that $|A'| = |A \cap [p]| \geq \alpha(1 - \frac{\alpha}{200})p$ since otherwise $|A \cap [p+1, N]| \geq \alpha(N-p) + \frac{\alpha^2 p}{200} \geq \alpha(1 + \frac{\alpha}{400})(N-p)$ so we would be in case (ii) with $J = [p+1, N]$.
Let $A'' = A' \cap \left[\frac{p}{3}, \frac{2p}{3}\right]$. Note that all 3-APs of the form $(x, x+d, x+2d) \in A' \times A'' \times A''$ are in fact proper APs in $[N]$.



Note that because distance is less than $p/3$, there's no wrapping around!
If $\left|A' \cap \left[\frac{p}{3}\right]\right|$ or $\left|A' \cap \left[\frac{2p}{3}, p\right]\right|$ are at least $\frac{2}{5}|A'|$ we are again in case (ii).
We may assume that $|A''| \geq \dfrac{|A'|}{5}$. Now, as in lemma 1.16 and theorem 1.17, with $\alpha' = \dfrac{|A'|}{p}$,

$\alpha'' = \dfrac{|A''|}{p}$ we have

$$\frac{\alpha''}{p} = \frac{|A''|}{p^2} = T_3(1_{A'}, 1_{A''}, 1_{A''}) = \alpha'(\alpha'')^2 + \sum_{t \neq 0} \widehat{1_{A'}}(t)\widehat{1_{A''}}(t)\overline{\widehat{1_{2 \cdot A''}}(t)}.$$

So as before $\dfrac{\alpha'(\alpha'')^2}{2} \leq \sup_{t \neq 0} |\widehat{1_{A'}}(t)|\alpha''$ provided that $\dfrac{\alpha''}{p} \leq \dfrac{\alpha'(\alpha'')^2}{2}$ which holds by assumption. $\square$

Lecture 5

We must convert the large Fourier coefficient into a density increment.

Lemma : 1.22

Let $m \in \mathbb{N}$ and let $\phi : [m] \to \mathbb{Z}_p$ taking $x \mapsto xt$ for some fixed $t \neq 0$. Given $\varepsilon > 0$ $\exists$ partition of $[m]$ into progressions P_i of length in $\left[\varepsilon \dfrac{\sqrt{m}}{2}, \varepsilon\sqrt{m}\right]$ such that $\mathrm{diam}(\phi(P_i)) = \max_{x,y \in P_i} |\phi(x) - \phi(y)| \leq \varepsilon p$ for all i.

Proof. Let $u = \lfloor \sqrt{m} \rfloor$ and consider $0, t, 2t, \ldots, ut$. By the pigeonhole principle we can find $0 \leq v < w \leq u$ such that $|wt - vt| \leq p/u$.
Divine $[m]$ into residue classes mod s, where $s = w - v$ (so $|st| \leq p/u$). Each of size at least $m/s \geq m/u$. But each residue class can be divided into progressions of the form $a, a+s, a+2s, \ldots, a+ds$ with $\dfrac{\varepsilon u}{2} < d \leq \varepsilon u$. The diameter of the image of each progression under φ is $|dst| < \varepsilon p$. $\square$

Lemma : 1.23

Let $A \subset [N]$ of density α. Let $p \in \left[\frac{N}{3}, \frac{2N}{3}\right]$ and $A' = A \cap [p] \subset \mathbb{Z}_p$. Suppose there exists $t \neq 0$ such that $|\widehat{1_{A'}}(t)| \geq \dfrac{\alpha^2}{10}$. Then there exists a progression P of length at least $\dfrac{\alpha^2\sqrt{N}}{500}$ such that $|A \cap P| \geq \alpha(1 + \dfrac{\alpha}{80})|P|$.

Proof. Let $\varepsilon = \dfrac{\alpha^2}{40\pi}$, and use lemma 1.22 to partition $[p]$ into progressions P_i of length at least $\dfrac{\varepsilon\sqrt{p}}{2} \geq \dfrac{\alpha^2}{40\pi} \dfrac{\sqrt{\frac{N}{3}}}{2} \geq \dfrac{\alpha^2\sqrt{N}}{500}$.
The diameter $\phi(P_i) \leq \varepsilon p$. Fix one x_i from each P_i we have

$$\frac{\alpha^2}{10} \leq |\widehat{1_{A'}}(t)| = |\widehat{f_{A'}}(t)| = \frac{1}{p} \Big| \sum_i \sum_{x \in P_i} f_{A'}(x)\omega^{xt} \Big|.$$

We have

$$\begin{aligned}\frac{1}{p}\left|\sum_i \sum_{x\in P_i} f_{A'}(x)\omega^{xt}\right| &= \frac{1}{p}\left|\sum_i \sum_{x\in P_i} f_{A'}(x)\omega^{x_i t} \sum_i \sum_{x\in P_i} f_{A'}(x)(\omega^{xt}-\omega^{x_i t})\right| \\ &\le \frac{1}{p}\sum_i \left|\sum_{x\in P_i} f_{A'}(x)\right| + \frac{1}{p}\sum_i \sum_{x\in P_i} \underbrace{|f_{A'}(x)|}_{\le 1} 2\pi\varepsilon\end{aligned} \tag{1}$$

since $|t(x_i - x)| \le \varepsilon p$ for all $x \in P_i$. We have that

$$\frac{1}{p}\left|\sum_i \sum_{x\in P_i} f_{A'}(x)\omega^{xt}\right| \le \frac{1}{p}\sum_i \left|\sum_{x\in P_i} f_{A'}(x)\right| + \frac{\alpha^2}{20}.$$

So we have

$$\frac{1}{p}\sum_i \left|\sum_{x\in P_i} f_{A'}(x)\right| \ge \frac{\alpha^2}{20}.$$

Since $f_{A'}$ has mean 0, we have

$$\sum_i \left(\left|\sum_{x\in P_i} f_{A'}(x)\right| + \sum_{x\in P_i} f_{A'}(x)\right) \ge \frac{\alpha^2 p}{20}.$$

So there exists i such that $\left|\sum_{x\in P_i} f_{A'}(x)\right| + \sum_{x\in P_i} f_{A'}(x) \ge \dfrac{\alpha^2|P_i|}{40}$ and so $\sum_{x\in P_i} f_{A'}(x) \ge \dfrac{\alpha^2|P_i|}{80}$. □

We'll put these together to prove theorem 1.20 in the example sheet.

Behrend's Example 1.24. There exists a set $A \subset [N]$ containing no non-trivial 3-APs of size

$$|A| \ge C\exp(-c\sqrt{\log N})N$$

where c, C are absolute constants.

Definition 1.25. Let $\Gamma \subset \widehat{G}$ and $\rho > 0$. By the **Bohr set** $B(\Gamma,\rho)$ we mean $B(\Gamma,\rho) = \{x \in G : |\gamma(x) - 1| \le \rho \; \forall \gamma \in \Gamma\}$. We call $|\Gamma|$ the **rank** and ρ the **radius** of the Bohr set.

Example 1.26. When $G = \mathbb{F}_p^n$, $B(\Gamma,\rho) = \langle\Gamma\rangle^\perp$ for all $\rho < 1$ (for $p = 3$).

Lemma : 1.27

Let $\Gamma \subset \widehat{G}$ be of size d, and let $\rho > 0$. Then $|B(\Gamma,\rho)| \ge \left(\dfrac{\rho}{2\pi}\right)^d |G|$.

Proof. We'll see this in example sheet 2. □

Lemma : Bogolyubov's again

Given $A \subset \mathbb{Z}_p$ of density $\alpha > 0$, there exists $\Gamma \subset \widehat{\mathbb{Z}_p}$ of size at most $2\alpha^{-2}$ such that $B(\Gamma, \frac{1}{2}) \subset A + A - A - A$.

Lecture 6

Proof. Recall $1_A * 1_A * 1_{-A} * 1_{-A}(x) = \sum_{t \in \widehat{\mathbb{Z}_p}} |\widehat{1_A}(t)|^4 \omega^{-xt}$. Let $\Gamma = \mathrm{Spec}_{\sqrt{\alpha/2}}(1_A)$ and note that for all $x \in B(\Gamma, \frac{1}{2})$ and $t \in \Gamma$, $\cos(\frac{2\pi xt}{p}) > 0$. Hence $Re(\sum_{t \in \widehat{Z_p}} |\widehat{1_A}(t)|^4 \omega^{-xt}) = \underbrace{\sum_{t \in \Gamma} |\widehat{1_A}(t)|^4 \cos(\frac{2\pi xt}{p})}_{\geq \alpha^4} + \underbrace{\sum_{t \notin \Gamma} |\widehat{1_A}(t)|^4 \cos(\frac{2\pi xt}{p})}_{(1)}$. In absolute value we have

$$|(1)| \leq \sup |\widehat{1_A}(t)|^2 \sum |\widehat{1_A}(t)|^2 \leq (\sqrt{\frac{\alpha}{2}} \cdot \alpha)^2 \cdot \alpha = \frac{\alpha^4}{2}.$$

□

Chapter 2: Combinatorial Methods

Lecture 6

For now, let G be an abelian group. Given $A, B \subset G$. We defined $A \pm B = \{a \pm b : a \in A, b \in B\}$. If A and B are finite, then

$$\max\{|A|, |B|\} \leq |A \pm B| \leq |A| \cdot |B|.$$

(better bounds are available in certain settings)

Example 2.1. Let $V \leq \mathbb{F}_p^n$ be a subspace, then $V + V = V$. So $|V + V| = |V|$. In fact, if $A \subset \mathbb{F}_p^n$ such that $|A + A| = |A|$ then A must be a coset of a subspace.

Example 2.2. Let $A \subset \mathbb{F}_p^n$ be such that $|A + A| < \frac{3}{2}|A|$. Then $\exists V \leq \mathbb{F}_p^n$ such that $A \subset V$ and $|V| \leq \frac{3}{2}|A|$. We'll see this in example sheet 2 (check, it may be wrong)

Example 2.3. Let $A \subset \mathbb{F}_p^n$ be a set of linearly independent vectors. Then $A + A$ has size $\binom{|A|}{2}$. But $|A| \leq n$ (small!)

Let $A \subset \mathbb{F}_p^n$ be a set chosen at random with probability $p^{-\theta n}$ for some $\theta \in (\frac{1}{2}, 1]$. Then with high probability $|A + A| = (1 - o(1))\dfrac{|A|^2}{2}$.

Definition 2.4. Given finite sets $A, B \subset G$ we define the **Ruzsa distance** $d(A, B)$ between A and B by

$$d(A, B) = \log \left(\frac{|A - B|}{\sqrt{|A| \cdot |B|}} \right).$$

$d(A, B)$ is clearly non-negative and symmetric.

Lemma : 2.5 - (Ruzsa's Triangle Inequality)

Given finite sets $A, B, C \subset G$, we have:

$$d(A,C) \leq d(A,B) + d(B,C).$$

Proof. Observe that $|B| \cdot |A - C| \leq |A - B| \cdot |B - C|$. Indeed, writing each $d \in A - C$ as $d = a_d - c_d$ for some $a_d \in A$, $c_d \in C$. The map $\phi : B \times (A - C) \to (A - B) \times (B - C)$ via $(b, d) \mapsto (a_d - b, b - c_d)$. You can easily check that this is injective. The triangle inequality follows from the definition of d. $\square$

Definition 2.6. Given a finite set $A \subset G$ we write $\sigma(A) = \dfrac{|A+A|}{|A|}$ for the **doubling constant** and $\delta(A) = \dfrac{|A-A|}{|A|}$ for the **difference constant** of A.

Lemma 2.5 tells us for example that

$$d(A,A) \leq d(A,-A) + d(-A,A)$$

So

$$\log(\delta(A)) \leq 2\log(\sigma(A))$$

Therefore $\delta(A) \leq \sigma(A)^2$ or $|A - A| \leq \dfrac{|A+A|^2}{|A|}$.

Notation: Given $A \subset G$ and $l, m \in \mathbb{N}_0$. Write $lA - mA$ for the set

$$\underbrace{A + A + \cdots + A}_{l-\text{times}} - \underbrace{A - A \cdots - A}_{m-\text{times}}.$$

Theorem : 2.7 - Plünnecke's Inequality

Let $A, B \subset G$ be finite sets such that $|A + B| \leq K|A|$ for some $K > 0$. Then for any $l, m \in \mathbb{N}_0$, $|lB - mB| \leq K^{l+m}|A|$.

Proof. WLOG assume that $|A + B| = K|A|$. Choose a nonempty subset $A' \subset A$ such that the ratio $\dfrac{|A'+B|}{|A'|}$ is minimized, and call this minimal ratio K'. Then $|A' + B| = K'|A|$, $K' \leq K$ and $|A'' + B| \geq K'|A''|$ for $A'' \subset A$.

Claim: For any finite $C \subset G$, $|A' + B + C| \leq K'|A' + C|$. (finishing the proof in Lecture 7:)

We first show that $|A' + mB| \leq K'^m|A'|$ for all $m \in \mathbb{N}_0$. We do this by induction: $m = 0\checkmark$, $m = 1\checkmark$. Suppose $m > 1$ and the result holds for $m - 1$. By the claim with $C = (m-1)B$ we get

$$|A' + mB| = |A' + B + (m-1)B| \leq K'|A' + (m-1)B| \leq K'^m|A'| \;\; \checkmark$$

As in the proof of Ruzsa's triangle inequality, $|A'| \cdot |lB - mB| \leq |A' + lB| \cdot |A' + mB| \leq K'^l|A'| \cdot K'^m|A'|$ Therefore $|lB - mB| \leq K'^{l+m}|A'| \leq K^{l+m}|A|$.

We now prove the claim by induction on $|C|$.
$|C| = 1$ ✓ Suppose the claim holds for C and consider $C' = C \cup \{x\}$ for some $x \notin C$. Observe $A' + B + C' = (A' + B + C) \cup (A' + B + x)$ and in fact $A' + B + C' = (A' + B + C) \cup ((A' + B + x) \setminus (D + B + x))$ where $D = \{a \in A' : a + B + x \subset A' + B + C\}$. By definition of K', $|D + B| \geq K'|D|$ therefore

$$\begin{aligned} |A' + B + C'| &\leq |A' + B + C| + |(A' + B + x) \setminus (D + B + x)| \\ &\leq |A' + B + C| + |A' + B| - |D + B| \\ &\leq K'|A' + C| + K'|A'| - K'|D| \\ &= K'(|A' + C| + |A'| - |D|) \end{aligned}$$

We apply the same argument again, writing

$$A' + C' = (A' + C) \sqcup ((A' + x) \setminus (E + x))$$

where $E = \{a \in A' : a + x \in A' + C\} \subset D$. We conclude that $|A' + C'| = |A' + C| + |A'| - |E| \geq |A' + C| + |A'| - |D|$. So $|A' + B + C'| \leq K'(|A' + C| + |A'| - |D|) \leq K'|A' + C'|$ which concludes the proof of our theorem. □

Lecture 7

We are now in a position to generalize example 2.2.

Theorem : Frieman-Ruzsa Theorem 2.8.

Let $A \subset \mathbb{F}_p^n$ be such that $|A + A| \leq K|A|$ (i.e. $\sigma(A) \leq K$) for some $K > 0$. Then A is contained in a coset of a subspace $H \leq \mathbb{F}_p^n$ of size $|H| \leq K^2 p^{K^4} |A|$.

Proof. Choose $X \subset 2A - A$ maximal such that the translates $x + A$ for $x \in X$ are disjoint. X cannot be too large because for all $x \in X$, $x + A \subset 3A - A$ and by Plunnecke, $|3A - A| \leq K^4|A|$ but the translates $x + A$ for $x \in X$ are disjoint and each of size $|A|$ so

$$|X| \cdot |A| = |\cup_{x \in X}(x + A)| \leq |3A - A|$$

Therefore $|X| \leq K^4$. We now show that

$$2A - A \subset X + A - A \tag{$\star$}$$

Indeed, if $y \in 2A - A$ and $y \notin X$, then $(y + A) \cap (x + A) \neq \varnothing$ for some $x \in X$ by maximality of X. So $y \in X + A - A$. If $y \in X$ then it's clear. It follows by induction from $(\star)$ that for all $l \geq 2$

$$lA - A \subset (l - 1)X + A - A \tag{$\star\star$}$$

(since $lA - A = A + (l-1)A - A \overset{hi}{\subset} A + (l-2)X + A - A = (l-2)X + 2A - A \overset{(\star)}{\subset} (l-1)X + A - A$.)
Now let H be the subgroup of $\mathbb{F}_p^n$ generated by A, which we can write as

$$H \subset \bigcup_{l \geq 1} (lA - A) \overset{(\star\star)}{\subset} Y + A - A.$$

where Y is the subgroup generated by X. Then $|Y| \leq p^{|X|} \leq p^{K^4}$ so

$$|H| \leq |Y + A - A| = |Y| \cdot |A - A| \leq p^{K^4} K^2 |A|.$$

□

Lecture 8

Example 2.9. Let $A = H \cup R \subset \mathbb{F}_p^n$ where $H \leq \mathbb{F}_p^n$ is a subspace of dimension d with $k <<< d << n - K$ and R consists of $K - 1$ linearly independent vectors in $H^\perp$. Then $|A| = |H \cup R| \sim |H|$ and $|A+A| = |(H \cup R)+(H \cup R)| = |(H+H) \cup (H+R) \cup (R+R)| \sim K|H|$ but any subspace $V \leq \mathbb{F}_p^n$ containing A must have size $\geq p^{d+(K-1)} = |H| p^{K-1} \sim |A| p^{k-1}$, where the constant is exponential in K.

Conjecture : 2.10 Polynomial Frieman-Ruzsa

Let $A \subset \mathbb{F}_p^n$ such that $|A + A| \leq K|A|$. Then there is a subspace $H \leq \mathbb{F}_p^n$ of size at most $C_1(K) \leq A$ such that for some $x \in \mathbb{F}_p^n$, $|A \cap (x + H)| \geq \frac{|A|}{C_2(K)}$, where $C_1(K)$ and $C_2(K)$ are polynomial in K.

For $p = 2$, this is now a theorem.

Definition 2.11. Given an abelian group G and finite sets $A, B \subset G$, define the **additive energy** between A and B to be

$$E(A, B) = \frac{\#\{(a, a', b, b') \in A \times A \times B \times B : \ a + b = a' + b'\}}{|A|^{3/2} |B|^{3/2}}.$$

We refer to quadruples $(a, a', b, b') \in A \times A \times B \times B$ such that $a + b = a' + b'$ as **additive quadruples**.

Observe that if G is finite, then

$$|A|^3 E(A, A) = |G|^3 \mathop{\mathbb{E}}_{x+y=z+w} [1_A(x) 1_A(y) 1_A(z) 1_A(w)] = |G|^3 \|\widehat{1_A}\|_4^4$$

This comes from example sheet 1.

Example 2.12. When $H \leq \mathbb{F}_p^n$, then $E(V, V) = 1$

Lemma : 2.13

Let G be abelian and let $A, B \subset G$ be finite. Then $E(A, B) \geq \dfrac{\sqrt{|A| \cdot |B|}}{|A + B|}$.

Proof. Note that

$$|A|^{3/2} |B|^{3/2} E(A, B) = \#\{(a, a', b, b') \in A^2 \times B^2 : a + b = a' + b'\} = \sum_{x \in G} r_{A+B}(x)^2.$$

where $r_{A+B}(x) = $ #ways of writing x as a+b with $a \in A, b \in B$. Observe that $\sum_{x \in G} r_{A+B}(x) = |A| \cdot |B|$. So

$$|A|^{3/2}|B|^{3/2}E(A,B) = \sum_{x \in G} r_{A+B}(x)^2 \geq \frac{(\sum_{x \in G} r_{A+B}(x))^2}{\sum_{x \in G} 1_{A+B}(x)} = \frac{(|A| \cdot |B|)^2}{|A+B|}.$$

Therefore

$$E(A,B) \geq \frac{\sqrt{|A| \cdot |B|}}{|A+B|}.$$

□

In particular if $A \subset G$ such that $|A+A| \leq K|A|$, then $E(A) \geq \frac{1}{K}$. The converse is NOT true.

Example 2.14. Let G be your favorite class of abelian group. Then there exists constants $\eta, \theta > 0$ such that for all sufficiently large n, there exists $A \subset G$ with $|A| = n$ satisfying $E(A,A) \geq \eta$ and $|A+A| \geq \theta |A|^2$. We'll see this in ExSheet2.

Theorem : 2.15 (Balog-Szemeredi-Gowers)

Let G be an abelian group, and let $A \subset G$ be finite such that $E(A,A) \geq \eta$ for some $\eta > 0$. Then $\exists A' \subset A$ of size at least $c(\eta)|A|$ such that $|A' + A'| \leq C(\eta)|A|$ where $c(\eta)$ and $C(\eta)$ are polynomial in η.

We first prove a technical lemma, using a method known as "dependent random choice".

Lemma : 2.16

Let $A_1, A_2, \ldots, A_m \subset [n]$ and suppose that $\sum_{i,j} |A_i \cap A_j| \geq \delta^2 n m^2$. Then there exists $X \subset [m]$ of size at least $\frac{\delta^5 m}{\sqrt{2}}$ such that $|A_i \cap A_j| \geq \frac{\delta^2 n}{2}$ for at least 90% of pairs $(i,j) \in X^2$.

Proof. Let x_1, x_2, x_3, x_4, x_5 be random from $[n]$, and let $X = \{i \in [m] : x_j \in A_i \ \forall j \in [5]\}$. Observe that if $|A_i \cap A_j| = \gamma n$, then $\mathbb{P}((i,j) \in X^2) = \gamma^5$ and hence (by convexity)

$$\mathbb{E}|X^2| = \sum_{i,j} \mathbb{P}((i,j) \in X^2) \geq \delta^{10} m^2.$$

Let us call a pair (i,j) "bad" if $|A_i \cap A_j| < \frac{\delta^2 n}{2}$. As before, $\mathbb{E}(\#\text{bad pairs in } X^2) \leq \frac{\delta^{10}}{2^5} m^2$. Hence $\mathbb{E}(|X^2| - 16\#\text{bad pairs in } X^2) \geq \frac{\delta^{10}}{2^5} m^2$. So there must be a choice of x_1, x_2, x_3, x_4, x_5 such that $|X| \geq \frac{\delta^5 m}{\sqrt{2}}$ and the proportion of bad pairs in X^2 is at most $\frac{1}{16} < 10\%$. □

Lecture 9

Proof of BSG. We call a difference d "popular" if d can be written as $d = x - y$ with $x, y \in A$ in at least $\eta\frac{|A|}{2}$ ways. i.e. $r_{A-A}(d) \geq \eta\frac{|A|}{2}$.
There must be at least $\eta\frac{|A|}{2}$ popular differences, because if not,

$$\eta|A|^3 \leq \sum_d r_{A-A}(d)^2 = \sum_{d\text{-pop}} r_{A-A}(d)^2 + \sum_{d\text{-unpop}} r_{A-A}(d)^2 < \eta\frac{|A|}{2}|A|^2 + \eta\frac{|A|}{2}\sum_{d\text{-unpop}} r_{A-A}(d)$$

So

$$\eta|A|^3 < \eta\frac{|A|}{2}|A|^2 + \eta\frac{|A|}{2}|A-A| \leq \eta\frac{|A|}{2}|A|^2 + \eta\frac{|A|}{2}|A|^2.$$

and this gives a contradiction.
Define a graph with vertex set A, joining x and y by an edge if and only if $y - x$ is a popular difference. Then $\mathop{\mathbb{E}}_{x\in A}[|N(x)|] = \frac{1}{|A|}\sum_{x\in A}|\underbrace{N(x)}_{\#y:y\sim x}| \geq \frac{\eta|A|}{2}$. We can also have $\mathop{\mathbb{E}}_{x,y\in A}|N(x) \cap N(y)| \geq \frac{\eta^2|A|}{4}$. Indeed,

$$\begin{aligned}\mathop{\mathbb{E}}_{x,y\in A}[|N(x)\cap N(y)|] &= \mathop{\mathbb{E}}_{x,y\in A}\left[\sum_{z\in A} 1_{N(x)}(z)1_{N(y)}(z)\right] = \sum_{z\in A}\left(\mathop{\mathbb{E}}_{x\in A} 1_{N(x)}(z)\right)^2 \\ &\geq \frac{1}{|A|}\left(\sum_{z\in A}\mathop{\mathbb{E}}_{x\in A} 1_{N(x)}(z)\right)^2 = \frac{1}{|A|}\left(\mathop{\mathbb{E}}_{x\in A}|N(x)|\right)^2 \\ &\geq \frac{1}{|A|}\left(\frac{\eta|A|}{2}\right)^2.\end{aligned}$$

We apply lemma 2.16 with $m = n = |A|$ and $\delta^2 = \frac{\eta^2}{4}$ to find a subset $A' \subset A$ of size $\geq \eta^{10}\frac{|A|}{2^{11}}$ with the property that $|N(x)\cap N(y)| \geq \frac{\eta^2|A|}{8}$ for at least 90% of $(x, y) \in A'^2$. But then for at least 10% of $x \in A'$, $|N(x) \cap N(y)| \geq \frac{\eta^2|A|}{8}$ for at least 80% of $y \in A'$. Hence there exists $A'' \cap A'$ of size $\geq \eta^{10}\frac{|A|}{2^{15}}$ such that for all $x \in A''$ at least 80% of $z \in A'$ satisfy $|N(x) \cap N(z)| \geq \frac{\eta^2|A|}{8}$.
Let $x, y \in A''$ then there are at least $\frac{\eta^{10}|A|}{2^{12}}$ many $z \in A'$ such that $|N(x) \cap N(y)| \geq \frac{\eta^2|A|}{8}$ and $|N(y) \cap N(z)| \geq \frac{\eta^2|A|}{8}$. We shall prove an upper bound on $|A'' - A''|$ by showing that each element of $A'' - A''$ can be written as a linear combination of distinct octuples from A. For each such z, there are at least $\left(\frac{\eta^2|A|}{8}\right)^2$ pairs (u, v) such that $u \in N(x) \cap N(y)$ and

$v \in N(y) \cap N(z)$.
For each such pair (u, v), the elements $u - x, z - u, v - z, y - v$ are all popular differences. Hence, for each pair (u, v) there are at least $\left(\frac{\eta|A|}{2}\right)^4$ octuples $(a_1, a_2, \ldots, a_8) \in A^8$ such that $u - x = a_2 - a_1$, $v - z = a_6 - a_5$, $z - u = a_4 - a_3$, $y - v = a_8 - a_7$. In other words, there are at least

$$\Big(\underbrace{\frac{\eta^{10}|A|}{2^{12}}}_{z}\Big) \cdot \Big(\underbrace{\frac{\eta^2|A|}{8}}_{u,v}\Big) \cdot \Big(\underbrace{\frac{\eta|A|}{2}}_{a_1,\ldots,a_8}\Big) = \frac{\eta^{18}}{2^{22}}|A|^7$$

octuples $(a_1, \ldots, a_8) \in A^8$ such that $y - x = \underbrace{a_2 - a_1}_{u-x} + \underbrace{a_4 - a_3}_{z-u} + \underbrace{a_6 - a_5}_{v-z} + \underbrace{a_8 - a_7}_{y-v}$. But distinct $y - x$ give rise to distinct octuples

$$\frac{\eta^{18}}{2^{22}}|A|^7 \cdot |A'' - A''| \leq |A|^8.$$

Hence

$$|A'' - A''| \leq 2^{22}\eta^{-18}|A| \leq 2^{27}\eta^{-28}|A''|.$$

$|A'' + A''|$ follows from Plunnecke. □

Chapter 3: Probabilistic Tools

Lecture 9

Proposition : 3.1 (Khintchine's inequality)

Let $X_1, X_2, \ldots, X_n$ be independent random variables, taking values $\pm x_i$ with probability $1/2$ for all i. Then for all $p \in [2, \infty)$ we have

$$\left\|\sum_{i=1}^n X_i\right\|_{L^p(\mathbb{P})} = O\left(p^{1/2}\left(\sum_{i=1}^n \|X_i\|^2_{L^2(\mathbb{P})}\right)^{1/2}\right).$$

The constant doesn't depend on p.

Proof. By nesting of norms, it suffices to prove the case $p = 2k$ with $k \in \mathbb{N}$. Write $X = \sum_{i=1}^n X_i$ and wlog assume that $\sum_{i=1}^n \|X_i\|^2_\infty = \sum_{i=1}^n \|X_i\|_2^2 = 1$. By Chernoff (example 1.3) $\forall\theta \geq 0$, $\mathbb{P}(|X| \geq \theta) \leq 4\exp(-\theta^2/4)$ so

$$\|X\|^{2k}_{L2k(\mathbb{P})} = \int_0^\infty 2kt^{2k-1}\mathbb{P}[|X| \geq t]dt \leq 8k \underbrace{\int_0^\infty t^{2k-1}\exp(-t^2/4)dt}_{=I(k)}.$$

We shall prove by induction on k that $I(k) \leq \dfrac{C^{2k}(2k)^k}{4k}$.
If $k = 1$, then $\displaystyle\int_0^\infty t\exp\left(-\frac{t^2}{4}\right)dt = \left[-2\exp(-t^2/4)\right]_0^\infty = 2 \leq \frac{C^2 \cdot 2}{4}$ if $C \geq 2$.

For $k > 1$, doing integration by parts yields

$$\begin{aligned}
I(k) &= \int_0^\infty t^{2k-2} \cdot t \exp(-t^2/4) dt \\
&= \left[t^{2k-2}(-2)\exp(-t^2/4)\right]_0^\infty - \int_0^\infty (2k-2)t^{2k-3}(-2)\exp(-t^2/4) dt \\
&= 4(k-1)\int_0^\infty t^{2(k-1)-1}\exp(-t^2/4) \\
&= 4(k-1)I(k-1)
\end{aligned}$$

By the inductive hypothesis we have

$$I(k) \leq \frac{4(k-1)C^{2(k-1)}(2(k-1))^{k-1}}{4(k-1)}$$

If $C \geq \sqrt{2}$ we get the desired conclusion. □

Lecture 10

Corollary : 3.2 - Rudin's inequality

Let $\Lambda \subset \widehat{\mathbb{F}_2^n}$ be a linearly independent set and let $p \in [2, \infty)$. Then for all $\widehat{f} \in \ell^2(\Lambda)$ (i.e. $\widehat{f} : \Lambda \to \mathbb{C}$) we have

$$\left\| \sum_{\gamma \in \Lambda} \widehat{f}(\gamma)\gamma \right\|_{L^p(\mathbb{F}_2^n)} = O\left(\sqrt{p} \cdot \|\widehat{f}\|_{\ell^2(\Lambda)}\right).$$

Corollary : 3.3 - Dual form of Rudin's inequality

Let $\Lambda \subset \widehat{\mathbb{F}_2^n}$ be a linearly independent set and let $p \in (1, 2]$ then for all $f \in L^p(\mathbb{F}_2^n)$,

$$\left\|\widehat{f}\right\|_{\ell^2(\Lambda)} = O\left(\sqrt{\frac{p}{p-1}} \cdot \|f\|_{L^p(\mathbb{F}_2^n)}\right)$$

Proof. Let $f \in L^p(\mathbb{F}_2^n)$ and write $g = \sum_{\gamma \in \Lambda} \widehat{f}(\gamma)\gamma$. Then

$$\|\widehat{f}\|^2_{\ell^2(\Lambda)} = \sum_{\gamma \in \Lambda} \left|\widehat{f}(\gamma)\right|^2 \stackrel{?}{=} \sum_{\gamma \in \Lambda} \widehat{f}(\gamma)\overline{\widehat{g}(\gamma)} = \langle \widehat{f}, \widehat{g} \rangle_{\ell^2(\Lambda)} = \langle \widehat{f}, \widehat{g} \rangle_{\ell^2(\mathbb{F}_2^n)}$$

By Plancharel, this is $\langle f, g \rangle_{L^2(\mathbb{F}_2^n)}$. By Holder, $\langle f, g \rangle_{L^2(\mathbb{F}_2^n)} \leq \|f\|_{L^p(\mathbb{F}_p^n)}\|g\|_{L^{p'}(\mathbb{F}_p^n)}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. By Rudin's inequality with p':

$$\|g\|_{L^{p'}(\mathbb{F}_2^n)} = O\left(\sqrt{p'}\|\widehat{g}\|_{\ell^2(\Lambda)}\right) = O\left(\sqrt{\frac{p}{p-1}}\|\widehat{f}\|_{\ell^2(\Lambda)}\right).$$

So

$$\|\widehat{f}\|_{\ell^2(\Lambda)}^2 = \|f\|_{L^p(\mathbb{F}_2^n)} O\left(\sqrt{\frac{p}{p-1}}\|\widehat{f}\|_{\ell^2(\Lambda)}\right)$$

The result follows after dividing on both sides by $\|\widehat{f}\|_{\ell^2(\Lambda)}$. □

Recall that given $A \subset \mathbb{F}_2^n$ of density $\alpha > 0$, $|Spec_\rho(1_A)| \leq \rho^{-2}\alpha^{-1}$. This is best possible, as the example of a subspace $H \leq \mathbb{F}_2^n$ shows: $Spec_1(1_H) = H^\perp$ so $|Spec_1(1_H)| = |H^\perp| = \dfrac{|\mathbb{F}_2^n|}{|H|} = \alpha^{-1}$.

Theorem : 3.4. - Special case of Chang's

Let $A \subset \mathbb{F}_2^n$ be a set of density $\alpha > 0$. Then for all $\rho > 0$, there exists a subspace $H \leq \mathbb{F}_2^n$ of dimension at most $O(\rho^{-2}\log(\alpha^{-1}))$ such that $H \supset Spec_\rho(1_A)$.

Proof. Let $\Lambda \subset Spec_\rho(1_A)$ be a maximal linearly independent subset of $Spec_\rho(1_A)$ and let $H = \langle Spec_\rho(1_A)\rangle$. Then $\dim(H) = |\Lambda|$. By corollary 3.3, $\forall p \in (1, 2]$,

$$|\Lambda|(\rho\alpha)^2 \leq \sum_{\gamma\in\Lambda} |\widehat{1_A}(\gamma)|^2 = \|\widehat{1_A}\|_{\ell^2(\Lambda)}^2 = O\left(\frac{p}{p-1}\|1_A\|_{L^p(\mathbb{F}_2^n)}^2\right).$$

We have $\|1_A\|_{L^p(\mathbb{F}_2^n)}^2 = \left(\mathbb{E}_y |1_A(y)|^p\right)^{2/p} = \alpha^{2/p}$. So $|\Lambda| \leq \rho^{-2}\alpha^{-2} O\left(\frac{p}{p-1}\alpha^{2/p}\right)$.
Choose $p = 1 + (\log(\alpha^{-1}))^{-1}$ to get $|\Lambda| = O(\rho^{-2}\log(\alpha^{-1}))$. □

Definition 3.5. Let G be a finite abelian group. We say $S \subset G$ is **dissociated** if $\sum_{s\in S} \varepsilon_s s = 0$ for some $\varepsilon_s \in \{-1, 0, 1\}^{|S|}$, then $\varepsilon \equiv 0$.
Note that if $G = \mathbb{F}_2^n$, then a set $S \subset G$ is dissociated if and only if it is linearly independent.

Lecture 11

Theorem : 3.6 - Chang's Theorem

Let G be a finite abelian group, and let $A \subset G$ of density $\alpha > 0$. If $\Lambda \subset Spec_\rho(1_A)$ is dissociated, then $|\Lambda| = O(\rho^{-2}\log(\alpha^{-1}))$.

We may bootstrap Khintchine's inequality to obtain the following:

Theorem : 3.7 - Marcinkiewicz-Zygmund inequality

Let $p \in [2, \infty)$, and let $X_1, X_2, \ldots, X_n \in L^p(\mathbb{P})$ be independent random variables with $\mathbb{E}\left[\sum_{i=1}^n X_i\right] = 0$. Then $\|\sum_{i=1}^n X_i\|_{L^p(\mathbb{P})} = O\left(p^{1/2}\left\|\sum_{i=1}^n |X_i|^2\right\|_{L^{p/2}(\mathbb{P})}^{1/2}\right)$.

Proof. For $\mathbb{C}$-valued random variables, the result follows from the real case by taking real and imaginary parts and applying the triangle inequality.
Next, assume the distribution of the X_i's is symmetric i.e. $\mathbb{P}(X_i = a) = \mathbb{P}(X_i = -a)$ for all $a \in \mathbb{R}$. Partition the probability space Ω into sets $\Omega_1, \Omega_2, \ldots, \Omega_M$ writing $\mathbb{P}_j$ for the induced measure on Ω_j such that all X_i's are symmetric and take at most 2 values on each Ω_j.
Applying Khintchine for each $j \in [M]$ we get

$$\left\|\sum_{i=1}^n X_i\right\|_{L^p(\mathbb{P}_j)}^p = O\left(p^{p/2}\left(\sum_{i=1}^n \|X_i\|_{L^2(\mathbb{P}_j)}^2\right)^{p/2}\right) = O\left(p^{p/2}\left\|\sum_{i=1}^n |X_i|^2\right\|_{L^{p/2}(\mathbb{P}_j)}^{p/2}\right).$$

So sum over all $j \in [M]$ and take the p-th root to get the symmetric case.
Now suppose X_i's are arbitrary and let $Y_1, \ldots, Y_n$ be such that $X_i \sim Y_i$ and $X_1, X_2, \ldots, X_n, Y_1, \ldots, Y_n$ are independent. Applying the symmetric result to $X_i - Y_i$ we get

$$\begin{aligned}\left\|\sum_{i=1}^n (X_i - Y_i)\right\|_{L^p(\mathbb{P}\times\mathbb{P})} &= O\left(p^{1/2}\left\|\sum_{i=1}^n |X_i - Y_i|^2\right\|_{L^{p/2}(\mathbb{P}\times\mathbb{P})}^{1/2}\right)\\ &= O\left(p^{1/2}\left\|\sum_{i=1}^n |X_i|^2\right\|_{L^{p/2}(\mathbb{P})}^{1/2}\right)\end{aligned}$$

But also
$$\left\|\sum_{i=1}^n X_i\right\|_{L^p(\mathbb{P})} = \left\|\sum_{i=1}^n X_i - \mathbb{E}\sum_{i=1}^n Y_i\right\|_{L^p(\mathbb{P})} \leq \left\|\sum_{i=1}^n (X_i - Y_i)\right\|_{L^p(\mathbb{P}\times\mathbb{P})}$$
by complexity. □

Theorem : 3.8 - Croot-Sisask Almost Periodicity

Let G be a finite abelian group, $\varepsilon > 0$, and $p \in [2, \infty)$. Let $A, B \subset G$ be such that $|A + B| \leq K|A|$ and let $F : G \to \mathbb{C}$. Then, there exists $b \in B$ and $X \subset B - b$ such that $|X| \geq (2K)^{-O(\varepsilon^{-2}p)}|B|$ and

$$\|\tau_x(f * \mu_A) - f * \mu_A\|_{L^p(G)} \leq \varepsilon\|f\|_{L^p(G)}$$

for all $x \in X$ where $\tau_x g(y) = g(y + x)$ and μ_A is the characteristic measure of A.

Proof. The main idea is to approximate $f * \mu_A(y) = \mathop{\mathbb{E}}_x \mu_A(x) f(y - x) = \mathop{\mathbb{E}}_{x\in A} f(y - x)$ by $\frac{1}{k}\sum_{i=1}^k f(y - Z_i)$ with Z_i sampled independently at random from A (say $Z = (Z_1, \ldots, Z_k)$), for some choice of k. For each $y \in G$ define $Z_i(y) = \tau_{-Z_i}(f)(y) - f * \mu_A(y)$. For fixed $y \in G$

these are independent and have mean zero, so by Marcinkiewicz-Zygmund,

$$\left\|\sum_{i=1}^k Z_i(y)\right\|_{L^p(\mathbb{P})}^p = O\left(p^{p/2}\left\|\sum_{i=1}^k |Z_i(y)|^2\right\|_{L^{p/2}(\mathbb{P})}^{p/2}\right)$$

$$= O\left(p^{p/2}\mathbb{E}\underbrace{\left|\sum_{i=1}^k |Z_i(y)|^2\right|^{p/2}}_{(1)}\right)$$

Using Holder with $\frac{2}{p}+\frac{1}{p'}=1$, we have

$$(1) \le \left(\sum_{i=1}^k 1^{p'}\right)^{1/p' \cdot /2}\left(\sum_{i=1}^k |Z_i(y)|^{2\cdot p/2}\right)^{2/p\cdot p/2} = k^{p/2-1}\sum_{i=1}^k |Z_i(y)|^p.$$

So for each $y \in G$,

$$\|\sum_{i=1}^k Z_i(y)\|_{L^p(\mathbb{P})}^p = O\left(p^{p/2}k^{p/2-1}\mathbb{E}\sum_{i=1}^k |Z_i(y)|^p\right).$$

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Summing over $y \in G$,

$$\mathop{\mathbb{E}}_{y\in G}\left\|\sum_{i=1}^k Z_i(y)\right\|_{L^p(\mathbb{P})}^p = O\left(p^{p/2}k^{p/2-1}\mathbb{E}\sum_{i=1}^k \mathop{\mathbb{E}}_{y\in G}|Z_i(y)|^p\right)$$

with $\left(\mathop{\mathbb{E}}_{y\in G}|Z_i(y)|^p\right)^{1/p} = \|Z_i\|_{L^p(\mathbb{P})} \le \underbrace{\|\tau_{-Z_i}(f)\|_{L^p(G)}}_{=\|f\|_{L^p(G)}} + \underbrace{\|f*\mu_A\|_{L^p(G)}}_{\le\|f\|_{L^p(G)}}$. Here we're using Young's convolution inequality: If $1+\frac{1}{r}=\frac{1}{q}+\frac{1}{p}$ then $\|f*g\|_r \le \|f\|_p\|g\|_q$. It follows that

$$\mathop{\mathbb{E}}_{Z\in A^k}\mathop{\mathbb{E}}_{y\in G}\left|\sum_{i=1}^k Z_i(y)\right|^p = O\left(p^{p/2}k^{p/2-1}\mathop{\mathbb{E}}_{Z\in A^k}\sum_{i=1}^k 2\|f\|_{L^p(G)}^p\right)$$

$$= O\left(p^{p/2}k^{p/2}\|f\|_{L^p(G)}^p\right)$$

$$= O\left((pk\|f\|_{L^p(G)}^2)^{\frac{p}{2}}\right)$$

This implies

$$\underbrace{\mathop{\mathbb{E}}_{Z\in A^k}\mathop{\mathbb{E}}_{y\in G}\left|\frac{1}{k}\sum_{i=1}^k [\tau_{-Z_i}(f)(y) - f*\mu_A(y)]\right|^p}_{(\star)} = O\left((pk^{-1}\|f\|_{L^p(G)}^2)^{\frac{p}{2}}\right).$$

Choose $k = O(\varepsilon^{-2}p)$ such that the RHS is at most $\left(\frac{\varepsilon}{4}\|f\|_{L^p(G)}\right)^p$. Write

$$L = \left\{(Z_1, \ldots, Z_k) \in A^k : (\star) \leq \left(\frac{\varepsilon}{2}\|f\|_{L^p(G)}\right)^p\right\}$$

By Markov since $\mathbb{E}(\star) \leq (\frac{\varepsilon}{2}\|f\|_{L^p(G)})^p = 2^{-p}\left(\frac{\varepsilon}{2}\|f\|_{L^p(G)}\right)^p$.

$$\frac{|L^c|}{|A^k|} = \mathbb{P}\left((\star) \geq (\frac{\varepsilon}{2}\|f\|_{L^p(G)})^p\right) \leq \mathbb{P}((\star) \geq 2^p\mathbb{E}(\star)) \leq 2^{-p}.$$

This implies that $\frac{|L|}{|A|^k} \geq 2^{-p}$ so in particular $|L| \geq \frac{1}{2}|A|^k$. Let $D = \{\underbrace{(b, b, \ldots, b)}_{k-\text{times}} : b \in B\}$, so $L + D \subset (A+B)^k$ thus

$$|L + D| \leq |(A+B)^k| \leq (K|A|)^k = k^k|A|^k \leq (2K)^k|L|$$

since $|L| \geq \frac{1}{2}|A|^k$.

By lemma 2.13, $E(L+D, L+D) \geq \frac{|D|^2|L|}{(2K)^k}$, so there are at least $\frac{|D|^2}{(2K)^k}$ pairs $(b_1, b_2) \in D \times D$ such that $r_{L-L}(b_1 - b_2) > 0$. In particular, there exists $b \in B$ and $X \subset X - b$ of size $|X| \geq \frac{|B|}{(2K)^k}$ such that $r_{L-L}(x) > 0$ for all $x \in X$. In other words, for all $x \in X$ there exist $l_1(x), l_2(x) \in L$ such that $\forall i \in [k]$, $l_1(x)_i = l_2(x)_i + x$ ($_i$ means i-th coordinate).

By the triangle inequality for each $x \in X$

$$\begin{aligned}
\|\tau_{-x}(f * \mu_A) - f * \mu_A\|_{L^p(G)} &\leq \left\|\tau_{-x}(f * \mu_A) - \tau_{-x}\left(\frac{1}{k}\sum_{i=1}^k \tau_{-l_2(x)_i}(f)\right)\right\|_{L^p(G)} \\
&\quad + \left\|\tau_{-x}\left(\frac{1}{k}\sum_{i=1}^k \tau_{-l_2(x)_i}(f)\right) - f * \mu_A\right\|_{L^p(G)} \\
&\leq \left\|f * \mu_A - \frac{1}{k}\sum_{i=1}^k \tau_{-l_2(x)_i}(f)\right\|_{L^p(G)} + \left\|\frac{1}{k}\sum_{i=1}^k \tau_{-x-l_2(x)_i}(f) - f * \mu_A\right\|_{L^p(G)} \\
&\leq 2\frac{\varepsilon}{2}\|f\|_{L^p(G)}
\end{aligned}$$

by the definition of L. □

Theorem : 3.9 - Bogolyubov, due to Sanders

Let $A \subset \mathbb{F}_p^n$ be a set of density $\alpha > 0$. Then there exists a subspace $V \leq \mathbb{F}_p^n$ of codimension $O(\log^4(\alpha^{-1}))$ such that $V \subset A + A - A - A$.

Proof. Ex Sheet 3. Chang & Croot-Sisask. □

Theorem : 3.10 - due to Schoen and Shkredov

Let $p \neq 5$ and $A \subset \mathbb{F}_p^n$. Suppose that A contains no non-trivial solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 5y$ i.e. no solutions such that $y \neq x_i$ for some $i \in [5]$. Then $|A| = \exp(-\Omega_p(\log |\mathbb{F}_p^n|^{\frac{1}{5}}))|\mathbb{F}_p^n|$.

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Proof. Let $\alpha = \dfrac{|A|}{|\mathbb{F}_p^n|}$ and partition A into $A_1 \cup A_2$ with $|A_1| = \left\lfloor \dfrac{\alpha}{2} p^n \right\rfloor$ and $|A_2| = \left\lceil \dfrac{\alpha}{2} p^n \right\rceil$. By averaging $\exists z \in \mathbb{F}_p^n$ such that $|A_1 \cap (z - A_2)| \geq \dfrac{\alpha^2}{4} p^n$. Let $A' = A_1 \cap (z - A_2)$. By theorem 3.9 there exists a subspace $V \leq \mathbb{F}_p^n$ of codimension $O(\log^4(\alpha^{-1}))$ such that $A' + A' - A' - A' \supset V$ and hence $2z + V \subset 2z + A' + A' - A' - A' \subset A_1 + A_1 + A_2 + A_2$.
Consequently, $(5 \cdot A - A) \cap (2z + V) = \varnothing$, for if there were $x, y \in A$ such that $5y - x \in 2z + V$, then we would be able to write $5y - x \in 2z + V$, then we would be able to write $5y - x$ as $a_1 + a_1' + a_2 + a_2'$ with $a_1, a_1' \in A_1$ and $a_2, a_2' \in A_2$ which since A_1 and A_2 are disjoint would yield a nontrivial solution.
It follows that for all $w \in \mathbb{F}_p^n$, at most one of $|A \cap (w + V)|$ and $5 \cdot A \cap (w + 2z + V)$ can be non-empty. Therefore, $2|A| = \sum_{w \in V^\perp} |A \cap (w+V)| + |5 \cdot A \cap (w + 2z + V)| \leq |V^\perp| \sup\limits_{w \in V^\perp} |A \cap (w + V)|$.
So there exists $w \in V^\perp$ such that $|A \cap (w + V)| \geq \dfrac{2|A|}{|V^\perp|} = \dfrac{2\alpha |\mathbb{F}_p^n|}{|\mathbb{F}_p^n|/|V|} = 2\alpha|V|$.
The set $A \cap (w + V) \subset w + V$ of density $\geq 2\alpha$, or equivalently $(A - w) \cap V \subset V$ of density $\geq 2\alpha$, containing no non-trivial solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 5y$.
After t iterations we obtain a subspace W of codimension $O(t \log^4(\alpha^{-1}))$ and $w \in \mathbb{F}_p^n$ such that $|A \cap (w + W)| \geq 2^t \alpha |W|$. Arguing as in the proof of theorem 1.17 yields the result. □

A similar bound in $\mathbb{Z}_N$ where Behrend's construction offers a comparable lower bound.

Chapter IV - Further Topics

Lecture 13

In $\mathbb{F}_p^n$ we can do much better, even for 3-APs.

Theorem : 4.1 (due to Ellenberg-Gijswijt based on Croot-Lev-Pach)

Let $A \subset \mathbb{F}_3^n$ be a set containing no non-trivial 3-APs. Then $|A| = o(2.765^n)$

Let M_n be the set of monomials in $x_1, \ldots, x_n$ whose degree in each variable is at most 2. Let V_n be the vector space over $\mathbb{F}_3$ generated by M_n. For any $d \in [0, 2n]$, write M_n^d for the set of monomials in M_n of (total) degree at most d, and V_n^d for the corresponding vector space. Set m_d for the dimension of V_n^d i.e. $|M_n^d|$.

Lemma : 4.2

Let $A \subset \mathbb{F}_3^n$ and suppose $P \in V_n^d$ is such that $P(a+a') = 0$ for all $a \neq a' \in A$. Then $|\{a \in A : P(2a) \neq 0\}| \leq 2m_{d/2}$.

Proof. Every $P \in V_n^d$ can be written as a linear combination of monomials from M_n^d, so $P(x+y) = \sum_{\deg(mm') \leq d} c_{m,m'} m(x) m'(y)$ for some coefficients $c_{m,m'}$.

Since at least one of m, m' has to have degree at most $\frac{d}{2}$, we can write $P(x+y) = \sum_{m \in M_n^{d/2}} m(x) F_m(y) + \sum_{m' \in M_n^{d/2}} m'(y) G_{m'}(x)$ where $(F_m)_{m \in M_n^{d/2}}$, $(G_{m'})_{m' \in M_n^{d/2}}$ are polynomials.
Viewing $(P(x+y))_{x,y \in A}$ as an $|A| \times |A|$ matrix C, we see that C can be written as a sum of at most $2m_{d/2}$ matrices of rank at most 1. Hence $\operatorname{rank}(C) \leq 2m_{d/2}$. But C is a diagonal matrix by assumption, whose rank equals $|\{a \in A : P(2a) \neq 0\}|$. $\square$

Proposition : 4.3

Let $A \subset \mathbb{F}_3^n$ be a set containing no non-trivial 3-APs. Then $|A| \leq 3m_{\frac{2n}{3}}$

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Proof. Let $d \in [1, 2n]$ to be chosen later. Let W be the subspace of V_n^d which vanish on $(2 \cdot A)^C$. Clearly, $\dim(W) \geq \dim(V_n^d) - |(2 \cdot A)^C| = m_d - (3^n - |2 \cdot A|)$.
Claim that there is $P \in W$ such that $|\operatorname{supp}(P)| \geq \dim(W)$. Indeed, pick $P \in W$ with maximal support. If $|\operatorname{supp}(P)| < \dim(W)$ then there would be a nonzero $Q \in W$ vanishing on $\operatorname{supp}(P)$, in which case $\operatorname{supp}(P+Q) \supsetneq \operatorname{supp}(P)$, contradicting our choice of P.
By assumption $\{a + a' : a \neq a' \in A\} \cap 2 \cdot A = \varnothing$. So any polynomial that vanishes $(2 \cdot A)^C$ also vanishes on $\{a + a' : a \neq a' \in A\}$.
By lemma 4.2 therefore

$$|\operatorname{supp}(P)| = |\{x \in \mathbb{F}_3^n : P(x) \neq 0\}| = |\{a \in A : P(2a) \neq 0\}| \leq 2m_{\frac{d}{2}}.$$

Putting everything together we have

$$m_d - (3^n - |A|) \leq \dim(W) \leq |\operatorname{supp}(P)| \leq 2m_{\frac{d}{2}}$$

thus $|A| \leq (3^n - m_d) + 2m_{\frac{d}{2}}$. But the monomials in $M_n \setminus M_n^d$ are in bijection with those of degree at most $2n - d$ (via $x_1^{\alpha_1} \dots x_n^{\alpha_n} \mapsto x_1^{2-\alpha_1} \dots x_n^{2-\alpha_n}$) thus $3^n - m_d = m_{2n-d}$. Thus setting $d = \frac{4n}{3}$ yields $|A| \leq 3m_{\frac{2n}{3}}$. $\square$

We'll deduce Theorem 4.1 on sheet 3. We do not know of a comparable bound for 4-APs. Fourier-analytic techniques also fail.

Example 4.4. Recall from lemma 1.16 that $|T_3(1_A, 1_A, 1_A) - \alpha^3| \le \sup_{t\neq 0}|\widehat{1_A}(t)|$.
But it is impossible to bound $|T_4(1_A, 1_A, 1_A, 1_A) - \alpha^4| = |\mathbb{E}_{x,d} 1_A(x)1_A(x+d)1_A(x+2d)1_A(x+3d) - \alpha^4|$ by $\sup_{t\neq 0}|\widehat{1_A}(t)|$. Indeed, consider $Q = \{x \in \mathbb{F}_p^n : x \cdot x = 0\}$. By problem 2 (ii) on sheet 1 we know that $\frac{|Q|}{p^n} = \frac{1}{p} + O(p^{-\frac{n}{2}})$ and $\sup_{t\neq 0}|\widehat{1_Q}(t)| = O(p^{-\frac{n}{2}})$.
But given a 3-AP $x, x+d, x+2d$ in Q, we automatically have that $x + 3d \in Q$.

$$\forall x, d \in \mathbb{F}_p^n,\ x \cdot x - 3(x+d)\cdot(x+d) + 3(x+2d)\cdot(x+2d) - (x+3d)\cdot(x+3d) = 0$$

So $T_4(1_A, 1_A, 1_A, 1_A) = T_3(1_A, 1_A, 1_A) = \alpha^3 + o(1)$.
Definition 4.5. Given $f : G \to \mathbb{C}$ with G finite abelian define its **U^2-norm** by the formula

$$\|f\|_{U^2(G)}^4 = \mathbb{E}_{x,a,b\in G} f(x)\overline{f(x+a)f(x+b)}f(x+a+b).$$

Problem 3(i) on sheet 1 showed that $\|f\|_{U^2(G)} = \|\widehat{f}\|_{\ell^4(G)}$, so this is indeed a norm. Problem 3(ii) asserted the following.

Lemma : 4.6

Let $f_1, f_2, f_3 : G \to \mathbb{C}$. Then $|T_3(f_1, f_2, f_3)| \le \min_{i\in[3]}\|f_i\|_{U^2(G)} \prod_{j\neq i}\|f_j\|_{L^\infty(G)}$.

Note that

$$\sup_{\gamma\in\widehat{G}}|\widehat{f}(\gamma)|^4 \le \sum_{\gamma\in\widehat{G}}|\widehat{f}(\gamma)|^4 \le \sup_{\gamma\in\widehat{G}}|\widehat{f}(\gamma)|^2 \sum_{\gamma\in\widehat{G}}|\widehat{f}(\gamma)|^2.$$

By Parseval, $\|\widehat{f}\|_{\ell^\infty(\widehat{G})} \le \|\widehat{f}\|_{\ell^4(\widehat{G})} = \|f\|_{U^2(G)} \le \|\widehat{f}\|_{\ell^\infty(G)}^{\frac{1}{2}} \cdot \|f\|_{L^2(G)}^{\frac{1}{2}}$.
Moreover, if $f = f_A = 1_A - \alpha$, then $T_3(f, f, f) = T_3(1_A - \alpha, 1_A - \alpha, 1_A - \alpha) = T_3(1_A, 1_A, 1_A) - \alpha^3$ plus three terms of the form $(-\alpha)\mathbb{E}_{x,d} 1_A(x+d)1_A(x+2d) = \alpha^3$ plus three terms of the form $(-\alpha)^2\mathbb{E}_{x,d} 1_A(x+3d) = \alpha^3$. So $T_3(f, f, f) = T_3(1_A, 1_A, 1_A) - \alpha^3$. We could therefore reformulate the first step in the proof of Meshulam's theorem as follows:
If $p^n \ge 2\alpha^{-2}$ then $\frac{\alpha^3}{2} \le |T_3(1_A, 1_A, 1_A) - \alpha^3| \le \|f_A\|_{U^2(G)}$ by lemma 4.6.

Lecture 15

Recasting theorem 1.17: IF $p^n \ge 2\alpha^{-2}$, then

$$\frac{\alpha^3}{2} \le \left|\frac{\alpha}{p^n} - \alpha^3\right| = |T_3(f_A, f_A, f_A)| \le \|f_A\|_{U^2}.$$

It remains to show that if $\|f_A\|_{U^2}$ is not too small then there exists a subspace $V \le \mathbb{F}_p^n$ of bounded codimension on A has increased density.

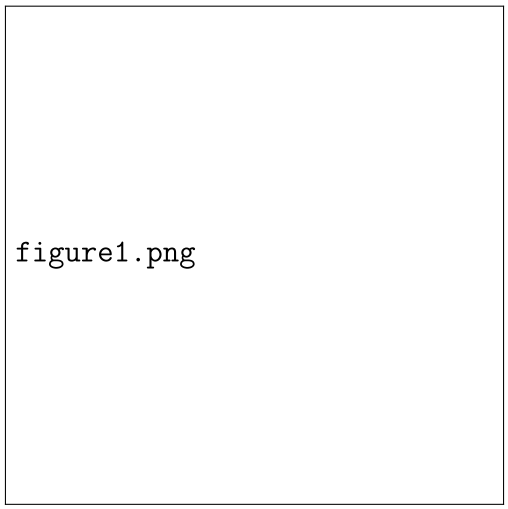
Theorem : 4.7 - U^2-Inverse

Let $f : \mathbb{F}_p^n \to \mathbb{C}$ satisfy $\|f\|_\infty \leq 1$ and $\|f\|_{U^2} \geq \delta$ for some $\delta \geq 0$. Then $\exists b \in \mathbb{F}_p^n$ such that $|\mathop{\mathbb{E}}_x f(x)\omega^{x \cdot b}| \geq \delta^2$.

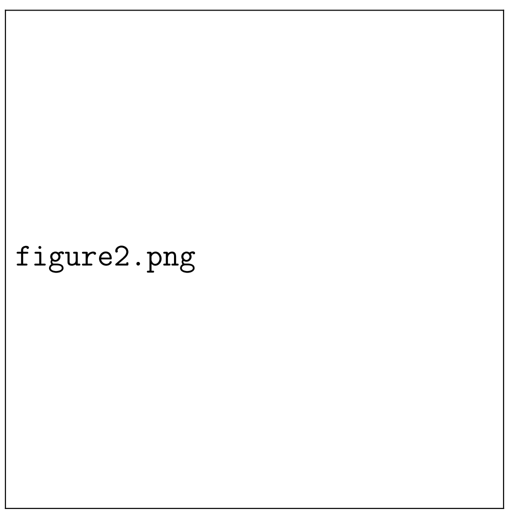
In other words, $|\langle f, \phi \rangle| \geq \delta^2$ for $\phi(x) = \omega^{x \cdot b}$ and we say "f correlates a linear function".

Proof. We've seen that $\|f\|_{U^2}^2 \leq \|\widehat{f}\|_{\ell^\infty} \cdot \|f\|_2 \leq \|\widehat{f}\|_{\ell^\infty}$, so $\delta^2 \leq \|\widehat{f}\|_{\ell^\infty} = \mathop{\mathbb{E}}_x f(x)\omega^{x \cdot b}$ for some $b \in \mathbb{F}_p^n$. $\square$

We can visualize the U^2 norm as a parallelogram:



We can extend this to the U^3 norm (soon to be defined) by adding an extra dimension:



Definition 4.8. Given $f : G \to \mathbb{C}$ with G finite abelian, define its **U^3-norm** by

$$\begin{aligned}\|f\|_{U^3(G)}^8 &= \mathop{\mathbb{E}}_{x,a,b,c \in G} f(x)\overline{f(x+a)f(x+b)f(x+c)}f(x+a+b)f(x+a+c)f(x+b+c)\overline{f(x+a+b+c)} \\ &= \mathop{\mathbb{E}}_{x,h_1,h_2,h_3} \prod_{\varepsilon \in \{0,1\}^3} \mathcal{C}^{|\varepsilon|} f(x + \varepsilon \cdot h).\end{aligned}$$

where $\mathcal{C}g(X) = \overline{g(x)}$ and $|\varepsilon| = \#1s$ in ε.
It's easy to verify that $\|f\|_{U^3(G)}^8 = \mathop{\mathbb{E}}_h \|\Delta_h f\|_{U^2(G)}^4$ where $\Delta_h f(x) = f(x)\overline{f(x+h)}$.

Definition 4.9. Given functions $f_\varepsilon : G \to \mathbb{C}$ for $\varepsilon = \{0,1\}^3$, define the **Gowers inner-product** by $\langle (f_\varepsilon)_{\varepsilon \in \{0,1\}^3} \rangle_{U^3(G)} = \mathop{\mathbb{E}}_{x,h_1,h_2,h_3} \prod_{\varepsilon \in \{0,1\}^3} \mathcal{C}^{|\varepsilon|} f_\varepsilon(x + \varepsilon \cdot h)$.
Observe that $\langle f, \ldots, f \rangle_{U^3(G)} = \|f\|_{U^3(G)}^8$.

Lemma : 4.10 - Gowers-Cauchy-Schwarz Inequality

Given $f_\varepsilon : G \to \mathbb{C}$ for $\varepsilon \in \{0,1\}^3$ $|\langle (f_\varepsilon)_{\varepsilon \in \{0,1\}^3} \rangle_{U^3(G)}| \leq \prod_{\varepsilon \in \{0,1\}^3} \|f_\varepsilon\|_{U^3(G)}$

Proof. ExSheet 3. □

Setting $f_\varepsilon = f$ for $\varepsilon \in \{0,1\}^2 \times \{0\}$ and $f_\varepsilon = 1$ otherwise. The LHS equals $\|f\|_{U^2(G)}^4$ so $\|f\|_{U^2(G)} \leq \|f\|_{U^3(G)}$.

Proposition : 4.11

Let $f : G \to \mathbb{C}$ with $\|f\|_{L^\infty(G)} \leq 1$. Then $|T_4(f, f, f, f)| \leq \|f\|_{U^3(G)}$.

Proof. It's long. Apply Cauchy-Schwarz 3 times. □

One might hope to generalize Meshulam's theorem as follows:

Theorem : 4.12 - Szemeredi's (for progressions of length 4)

Let $A \subset \mathbb{F}_p^n$ be a set containing no non-trivial 4-APs. Then $|A| = o(p^n)$.

Idea: By proposition 4.11 with $f = f_A$. $T_4(1_A, 1_A, 1_A, 1_A) - \alpha^4 = T_4(f_A, f_A, f_A, f_A)$ plus terms in which one and three of the inputs are equal to f_A, each of which is controlled $\|f_A\|_{U^2}$. Hence $|T_4(1_A, 1_A, 1_A, 1_A) - \alpha^4| \leq 14\|f_A\|_{U^3}$ since $\|\cdot\|_{U^2} \leq \|\cdot\|_{U^3}$. So if A contains no nontrivial 4-APs and $p^n \geq 2\alpha^{-3}$ then $\dfrac{\alpha^4}{2} \leq 14\|f_A\|_{U^3}$.

Lecture 16

What can we say about functions whose U^3-norm is large?
Example 4.13. Let M be an $n \times n$ (symmetric) matrix with entries in $\mathbb{F}_p$. Then $f(x) = \omega^{x^T M x}$ satisfies $\|f\|_{U^3} = 1$.

Theorem : 4.14 - U^3-Inverse Theorem

Let $f : \mathbb{F}_p^n \to \mathbb{C}$ satisfying $\|f\|_\infty \leq 1$ and $\|f\|_{U^3} \geq \delta$ for some $\delta > 0$. Then there exists a symmetric $n \times n$ matrix M with entries in $\mathbb{F}_p$ and $b \in \mathbb{F}_p^n$ such that $|\mathbb{E}_x f(x)\omega^{x^T Mx + b^T x}| \geq c(\delta)$ where $c(\delta)$ is a polynomial in δ (depending on p).

In other words, $|\langle f, \phi \rangle| \geq c(\delta)$ for $\phi(x) = \omega^{x^T Mx + b^T x}$ and we say "f correlates with a quadratic phase function".

Proof Sketch. Suppose $\|f\|_{U^3} \geq \delta$.

Step 1: "Weak linearity". If $\|f\|_{U^3}^8 = \mathbb{E}_h \|\Delta_h f\|_{U^2}^4 \geq \delta^8$ then for at least a $\frac{\delta^8}{2}$-proportion of $h \in \mathbb{F}_p^n$, $\|\Delta_h f\|_{U^2}^4 \geq \frac{\delta^8}{2}$, for each such $h \in \mathbb{F}_p^n$, $\exists t_n$ such that $|\widehat{\Delta_h f}(t_n)|^2 \geq \frac{\delta^8}{2}$. Working a tiny bit harder, one can obtain the following:

Proposition : 4.15

Let $f : \mathbb{F}_p^n \to \mathbb{C}$ satisfy $\|f\|_\infty \leq 1$ and $\|f\|_{U^3} \geq \delta$ for some $\delta > 0$. Suppose that $|\mathbb{F}_p^n| = \Omega_\delta(1)$. Then there exists a subset $S \subset \mathbb{F}_p^n$ with $\frac{|S|}{|\mathbb{F}_p^n|} = \Omega_\delta(1)$ and a function $\phi : S \to \mathbb{F}_p^n$ such that:

(i) $|\widehat{\Delta_h f}(\phi(h))| = \Omega_\delta(1)$.

(ii) There are at least $\Omega_\delta(|\mathbb{F}_p^n|^3)$ additive quadruples $(s_1, s_2, s_3, s_4) \in S^4$ and $\phi(s_1) + \phi(s_2) = \phi(s_3) + \phi(s_4)$.

Step 2: "Strong linearity"

If S and ϕ as above, then there's a linear map $\Psi : \mathbb{F}_p^n \to \widehat{\mathbb{F}_p^n}$ which coincides with ϕ for many elements of S. More precisely,

Proposition : 4.16

Let S and ϕ be given by Proposition 4.15. Then $\exists$ $n \times n$ matrix M with entries in $\mathbb{F}_p$ and $b \in \mathbb{F}_p^n$ such that the map $\psi : \mathbb{F}_p^n \to \widehat{\mathbb{F}_p^n}$ via $x \mapsto Mx + b$ satisfies $\Psi(x) = \phi(x)$ for $\Omega_\delta(|\mathbb{F}_p^n|)$ elements $x \in S$.

Proof. Consider the graph $\Gamma = \{(h, \phi(h)) : h \in S\} \subset \mathbb{F}_p^n \times \widehat{\mathbb{F}_p^n}$. By proposition 4.15, Γ has $\Omega_\delta(|\mathbb{F}_p^n|)$ additive quadruples. By the Balog-Szemeredi-Gowers theorem we have $\exists \Gamma' \subset \Gamma$ with $|\Gamma'| = \Omega_\delta(|\Gamma|) = \Omega_\delta(|\mathbb{F}_p^n|)$ and $|\Gamma' + \Gamma'| = O_\delta(|\Gamma'|)$.

Define S' by $\Gamma' = \{(h, \phi(h)) : h \in S'\}$ and note that $|S'| = \Omega_\delta(|\mathbb{F}_p^n|)$. By the Freiman-Ruzsa theorem applied to $\Gamma' \subset \mathbb{F}_p^n \times \widehat{\mathbb{F}_p^n}$, $\exists$ subspace $H \leq \mathbb{F}_p^n \times \widehat{\mathbb{F}_p^n}$ with $|H| = O_\delta(|\Gamma'|) = O_\delta(|\mathbb{F}_p^n|)$

such that $\Gamma' \subset H$.
Denote by $\pi : \mathbb{F}_p^n \times \widehat{\mathbb{F}_p^n} \to \mathbb{F}_p^n$ the projection onto the first n coordinates. By construction, $\pi(H) \supset S'$. Moreover, since $|S'| = \Omega_\delta(|\mathbb{F}_p^n|)$,

$$|\ker(\pi|_H)| = \frac{|H|}{|Im(\pi|_H)|} \leq \frac{O_\delta(|\mathbb{F}_p^n|)}{|S'|} = O_\delta(1).$$

We may thus partition H into $O_\delta(1)$ cosets of $H^* = \ker(\pi|_H)$ such that π is injective on each coset. By averaging, $\exists x + H^*$ such that $|\Gamma' \cap (x + H^*)| = \Omega_\delta(|\Gamma'|) = \Omega_\delta(|\mathbb{F}_p^n|)$. Set $\Gamma" = \Gamma' \cap (x + H^*)$ and define $S"$ accordingly.
Now $\pi|_{x+H^*}$ is both injective and surjective onto its image. $V = Im(\pi|_{x+H^*})$. But this means that $\exists$ affine linear map $\Psi : V \to \widehat{\mathbb{F}_p^n}$ such that $(h, \Psi(h)) \in \Gamma"$ for all $h \in S"$. $\square$

Step 3: "The symmetry argument"
Having obtained $\Psi(x) = Mx + b$ for some matrix M and vectors b such that $(h, Mh + b) \in \Gamma" \ \ \forall h \in S"$ we need to turn M into a symmetric matrix in preparation for step 4.
Step 4: "Integrating"